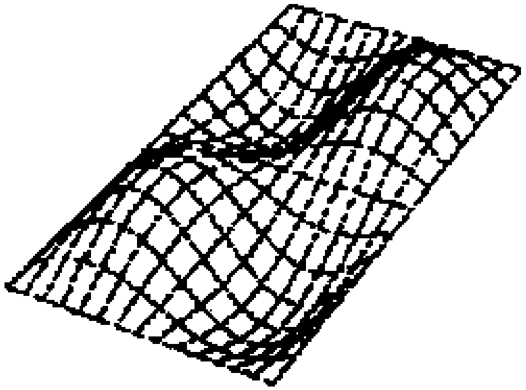
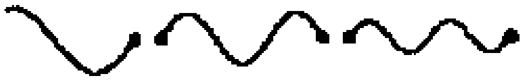


THE RECTANGULAR DRUM VS. THE STRING ("1Ddrum")



1) **How do their** temporal/circular frequencies differ? What are the **implications** of this regarding the musical quality of these **instruments?**

2) What do the individual modes look like?

3) What is the main complication in the solution process for these two systems?

THE RECTANGULAR DRUM VS. THE STRING ("1Ddrum")

(1) Circular frequency and period

Each component $u_n(x,t)$ is a PERIODIC function

of period $T = \dfrac{2\pi}{c\omega_n}$ *where $c\omega_n$ is the circular frequency*

$$DRUM: \quad c\omega_{mn} = c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$STRING: \quad c\omega_n = \frac{n\pi c}{L}$$

$$c = \sqrt{\frac{tension}{density}}$$

In contrast with the drum, the string vibrates at frequencies that are integer multiples of the FUNDAMENTAL frequency $c\omega_1 = \dfrac{\pi c}{L}$.

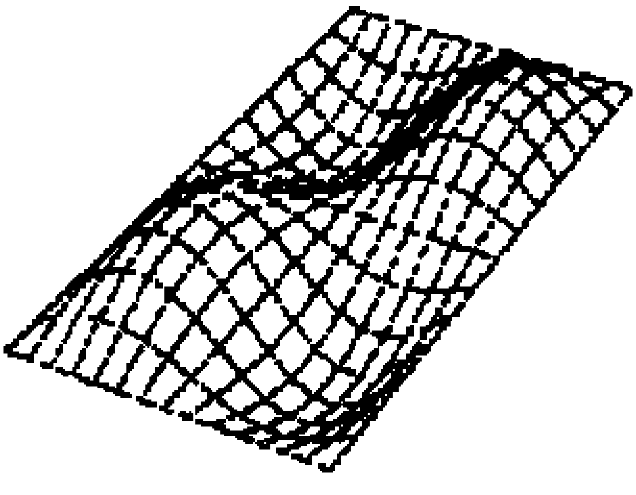
(2) Shape of Standing Waves

1D standing waves for string; 2D standing waves for drum

The shape of the "m,n MODE" is given by

$$\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

This is modulated periodically in time. If, say, the initial velocity is zero (i.e. $B_{mn} = 0$), then the "m,n MODE" is modulated by the $\cos\omega_{mn}ct$ factor.



(3) Where are the nodes?

Nodes can be found where the deflections $\boxed{u=0}$ for all time !

$$\sin\frac{m\pi x}{a}=0 \quad \mapsto \frac{m\pi x}{a}=\pi, 2\pi, 3\pi.......$$

$$\sin\frac{n\pi y}{b}=0 \quad \mapsto \frac{n\pi y}{b}=\pi, 2\pi, 3\pi.......$$

YOU CHECK that this occurs at

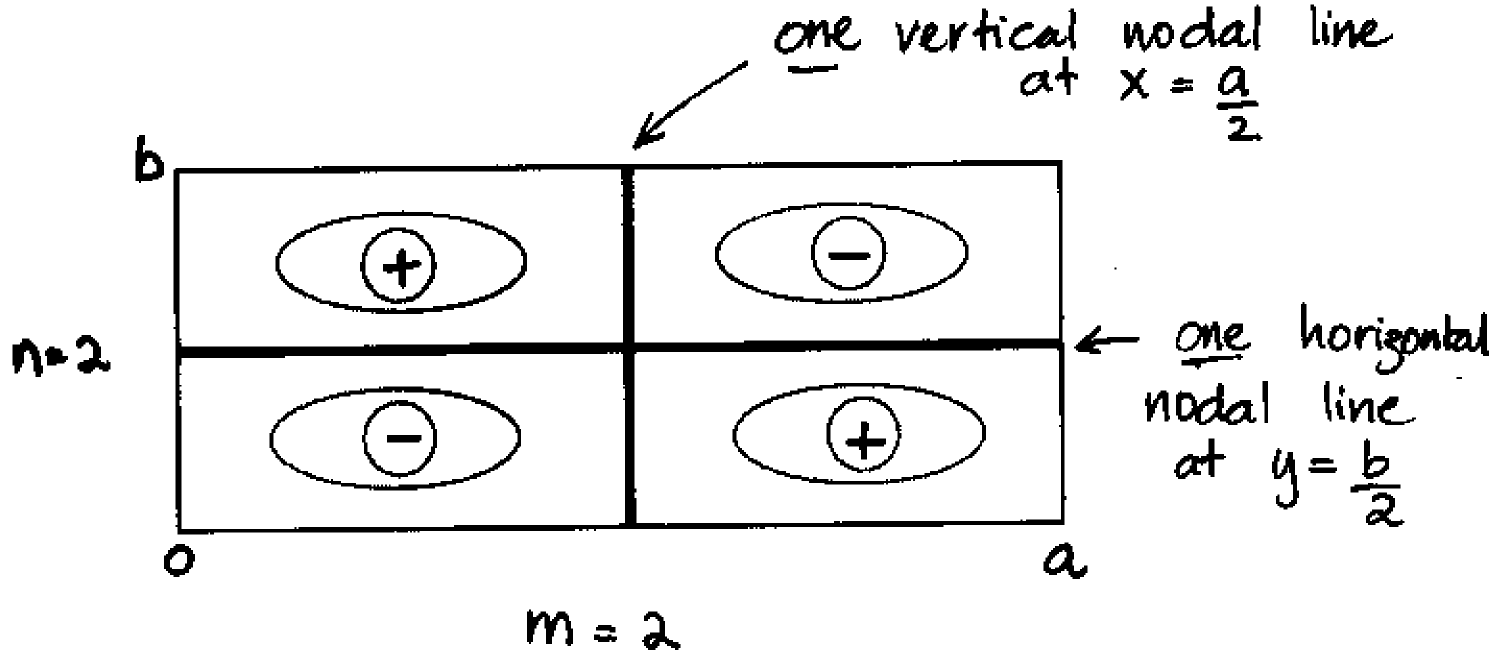
$$x=\ \frac{a}{m},\ \frac{2a}{m},\ ....,\ \frac{(m-1)a}{m}$$

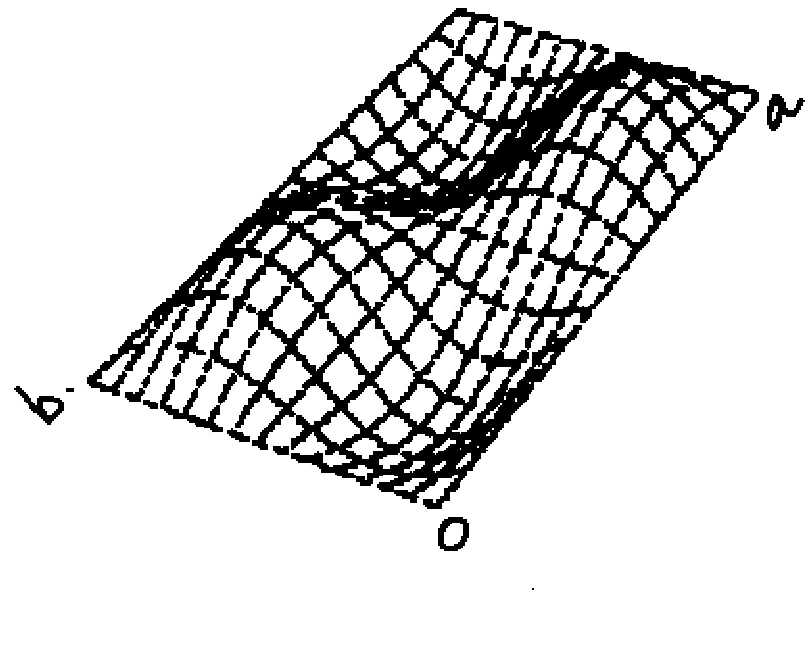
(m - 1) nodal lines

$$y=\ \frac{b}{n},\ \frac{2b}{n},\ ....,\ \frac{(n-1)b}{n}$$

(n - 1) nodal lines

For example, lets look at the (2,2) mode





Notes

tbh

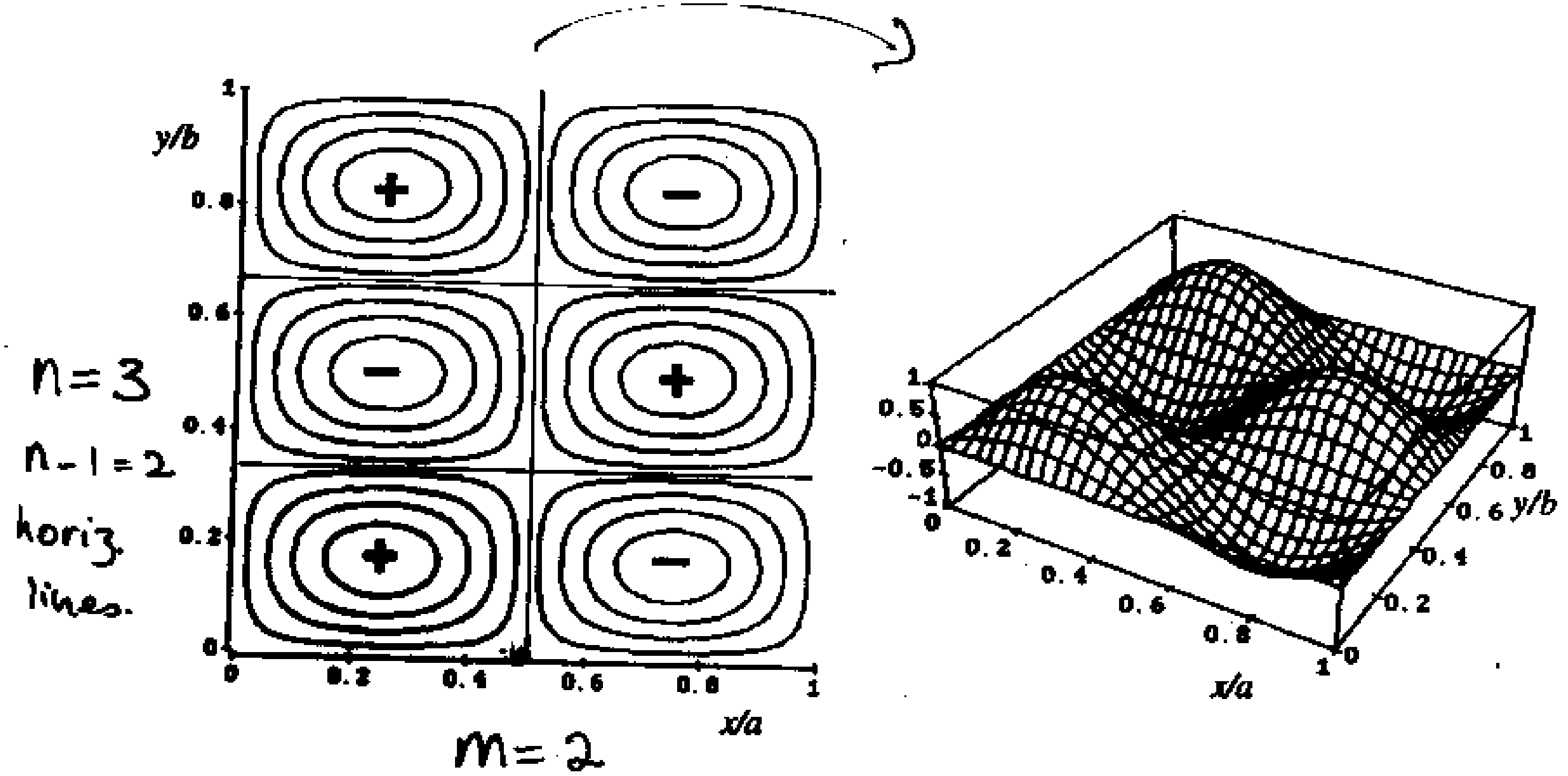


Figure 2.5: A contour plot and a perspective view of the standing wave $\sin(2\pi x/a)\sin(3\pi y/b)$.

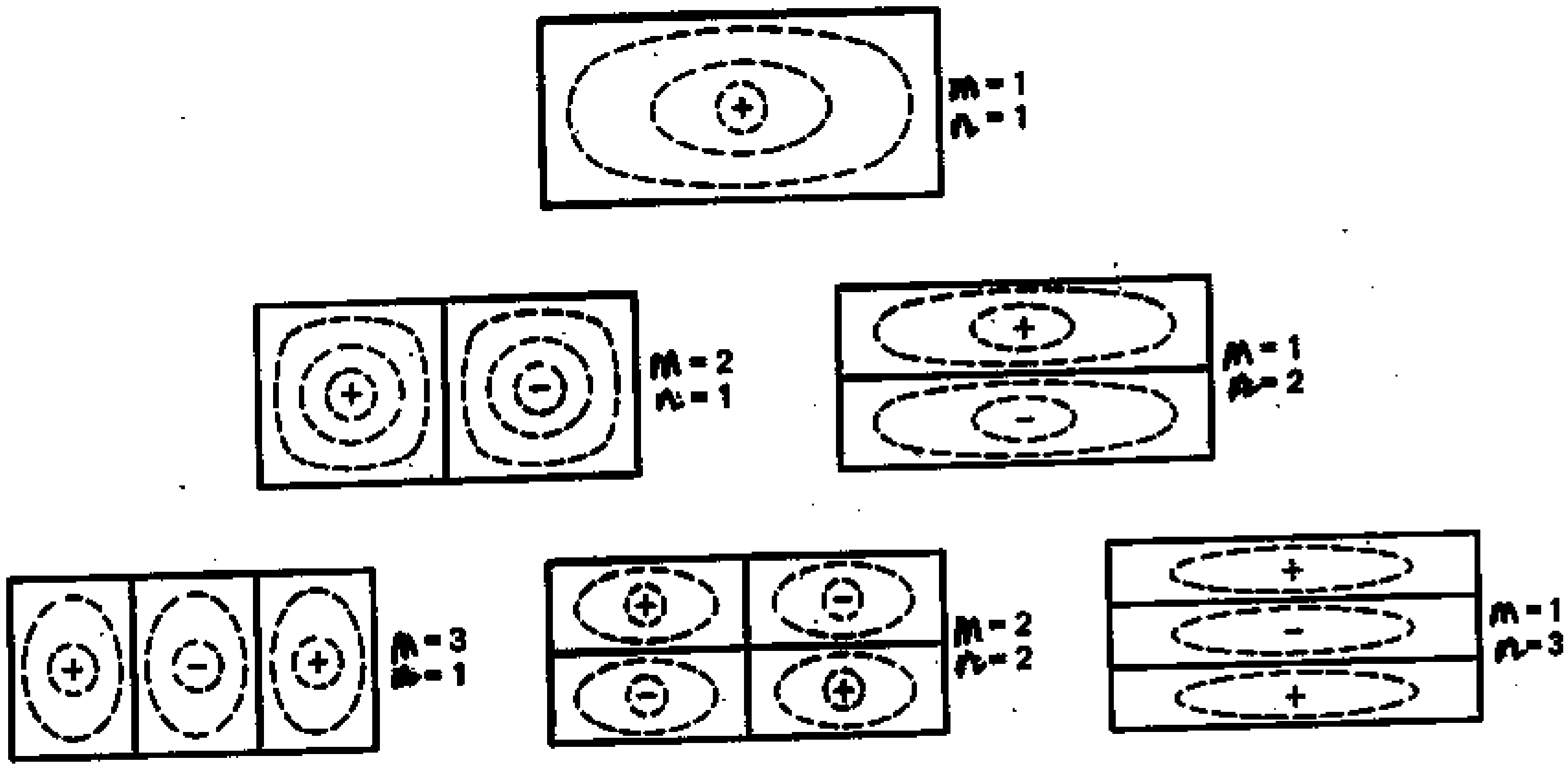
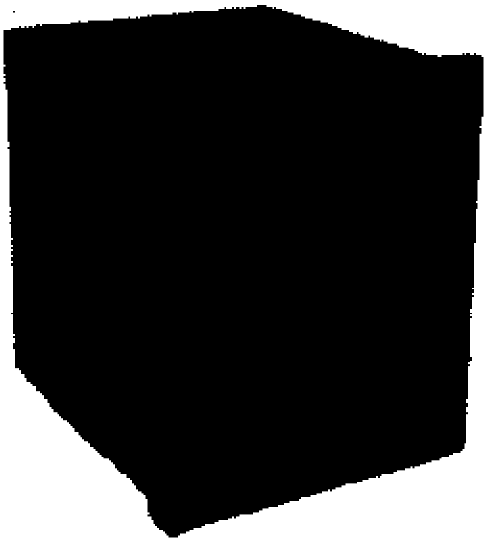


Figure 6.3.2 Nodal curves for modes of a vibrating rectangular membrane.

(4) What can our solution tell us about the musical quality of a string instrument (1D drum) vs a drum?



Suppose we tune the string (by adjusting the tension) so its fundamental frequency $\pi c/L$ corresponds to lowest A on the piano (Ao): i.e. 27.5 cycles/sec. This means adjust tension so that $T=(55L)^2\lambda$.

$$\underbrace{\frac{\pi c}{L}}_{\frac{rad}{sec}} = \frac{\pi}{L}\cdot\sqrt{\frac{T}{\lambda}} = 27.5\,\underbrace{(2\pi)}_{\frac{rad}{cycle}}$$

The first 5 notes in our general solution

n	Frequency $n\frac{\pi c}{L}$ (cycles/sec)	Relative Amplitude $\frac{1}{n^2}\left\|\sin\frac{n\pi}{2}\right\|$	Musical Note
1	27.5	1	A_0
2	55.0	0	A_1
3	82.5	$\frac{1}{9}$	$\approx E_2$
4	110.0	0	A_2
5	137.5	$\frac{1}{25}$	$\approx C_3^{\#}$

The sound is not a pristine A_0 with octave overtones, but is "fairly clean" due to the relatively small amplitudes of the E_2 + $C_3^{\#}$ contributions. The mix of frequencies and amplitudes is different for different instruments — an A_0 played on a violin sounds different from an A_0 on a tuba.

Relative amplitude for a rec. drum

$$\frac{64h}{\pi^6 m^3 n^3}$$

mn	ω_{mn}	Note
11	27.5	A_0
12,21	43.5	$\approx F_1$
22	55.0	A_1
13,31	61.5	$\approx B_1$
23,32	70.1	$\approx C_2^\#$
14,41	80.2	$\approx E_2$
33	82.5	$\approx E_2$
24,42	87.0	$\approx F_2$
34,43	97.2	$\approx G_2$
15,51	99.2	$\approx G_2$
25,52	104.7	$\approx G_2^\#$
44	110.0	A_2
35,53	113.4	$\approx A_2^\#$
16,61	118.3	$\approx A_2^\#$
26,62	123.0	$\approx B_2$
45,54	124.5	$\approx B_2$
36,63	130.4	$\approx C_3$
17,71,55	137.5	$\approx C_3^\#$
46,64	140.2	$\approx C_3^\#$

Figure 4. Square drum, octave overtones underlined.

$$W_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

Frequency profile for a drum (square) tuned so that its fundamental frequency is 27.5 cycles/sec. (i.e. A_0 on a piano)

Note that every note is present!

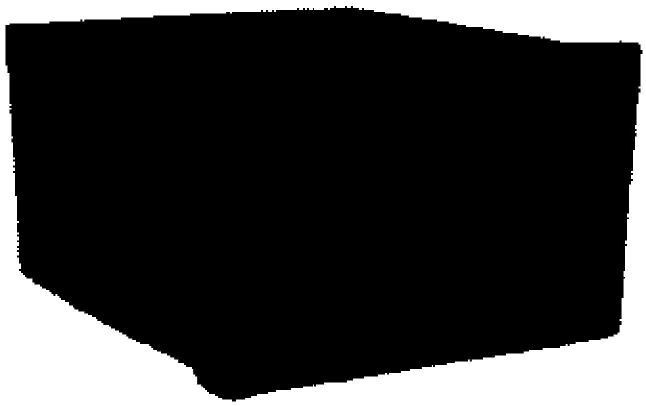
Compare this with the frequency profile for a string in a violin/guitar

FREQUENCY depends on the geometry

Why aren't rectangular drums prized instruments?

Virtually every note is present:

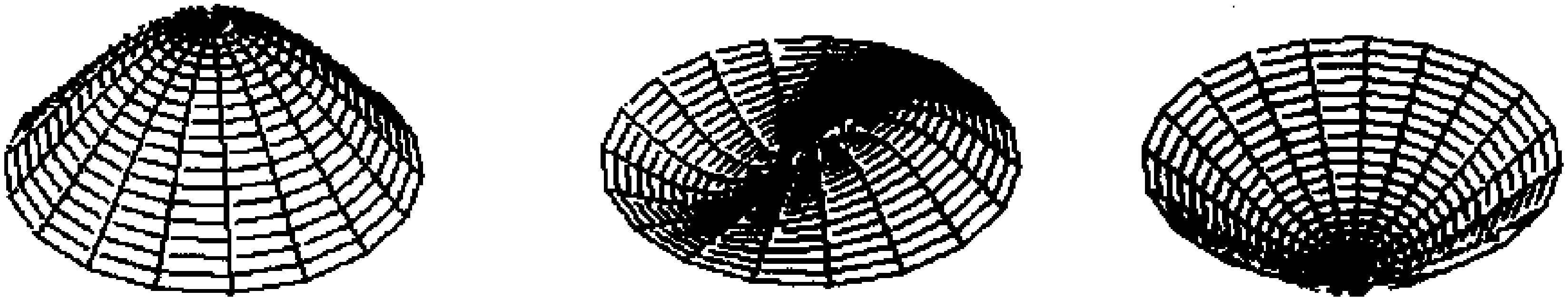
profusion of notes are due to the dense values of $\sqrt{m^2+n^2}$

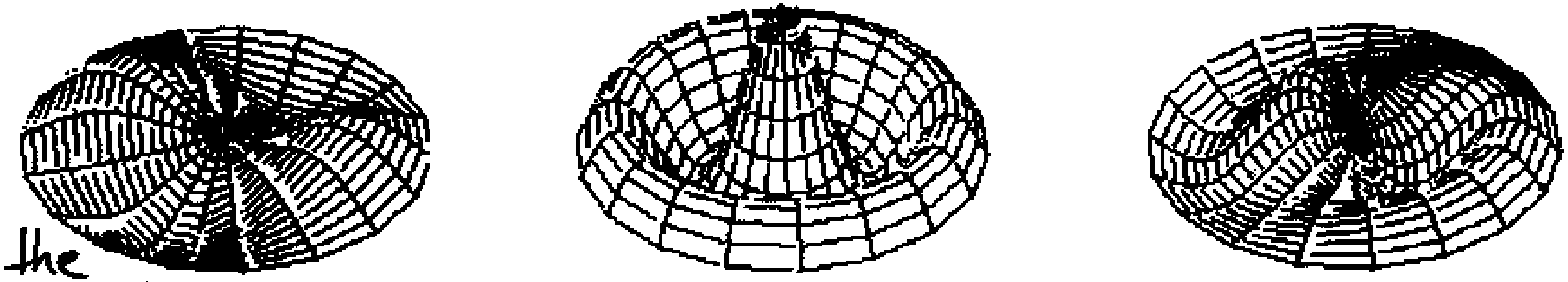


Playing it is like playing the piano with your forearms - rather than your fingers!!!!!

Circular drums are better!

What do the individual modes look like in a circular drum?

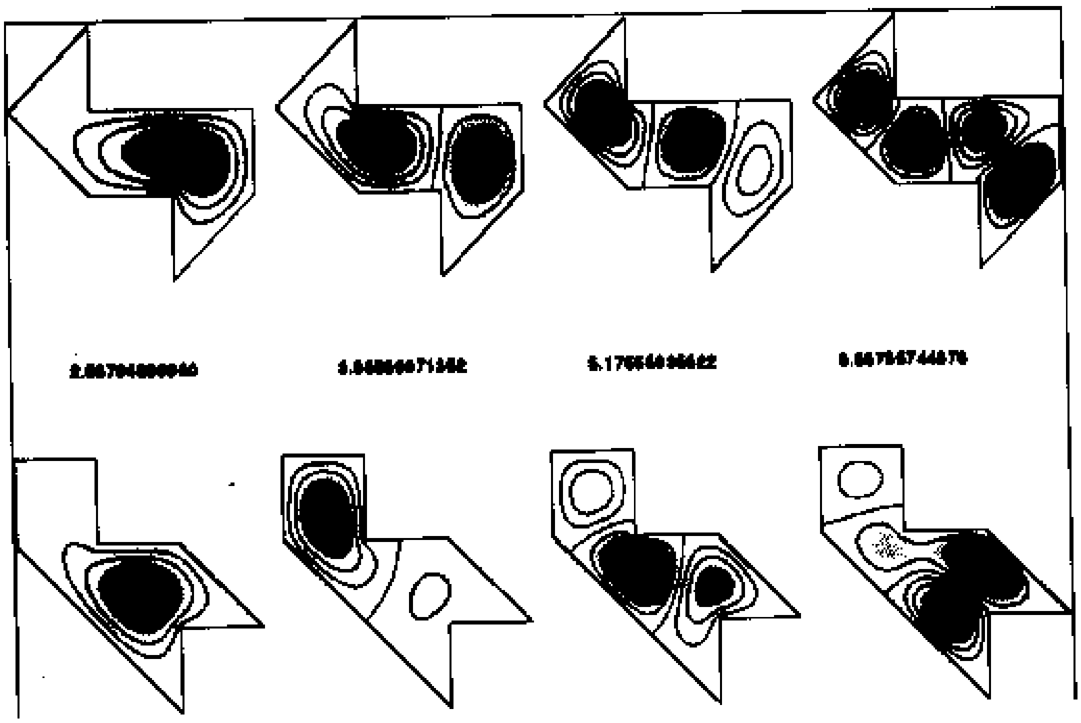




What do the modes look like if there is no θ-depencence? Can you guess!?

(above modes **with** θ-dependence)

In 1991, mathematicians Gordon, Webb, and Wolpert (Invent. Math. 110, pp 1--22) solved a famous problem posed by M. Kac in 1966: **"Can you hear the shape of a drum?"** That is, do the eigenvalues of a membrane determine the shape of the membrane? Their answer was "No!". They used a powerful mathematical technique to produce a counterexample, which in its simplest form is a pair of eight-sided nonconvex polygons.



2 different shape drums with identical frequency spectrum.